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A remark about dihedral group actions on spheres

Ian Hambleton

Abstract

We show that a finite dihedral group does not act pseudofreely and locally linearly on an even-dimensional sphere S^{2k}, with $k > 1$. This answers a question of Kulkarni from 1982.

1. *Introduction*

In this note, we let $D_p = \langle a, b \,|\, a^p = b^2 = 1,\ bab = a^{-1}\rangle$ denote the finite dihedral group of order $2p$, for p an odd prime. A famous theorem of Milnor [**8**] states that a finite dihedral group cannot act freely on a topological n-manifold with the mod 2 homology of S^n. More generally, a *pseudofree* action is one which is free outside of a discrete set of points. In [**6**, Theorem 7.4], Kulkarni studied orientation-preserving, pseudofree actions of finite groups G on manifolds which are $\mathbb{Z}/2$-homology n-spheres, and found that, for $n = 2k$, the group G must be (i) a periodic group which acts freely on S^{2k-1}, (ii) dihedral, or (iii) tetrahedral, octahedral, or icoshedral (when $k = 1$). The first case occurs as the suspension of any free action of a periodic group on S^{2k-1}, and the other cases already appear for orthogonal actions on S^2. Kulkarni asked whether the second case could actually occur on S^{2k} if $k > 1$. This turns out to be impossible.

Theorem A. *The dihedral group $G = D_p$, with p an odd prime, cannot act pseudofreely and locally linearly on S^{2k}, preserving the orientation, for $k > 1$.*

For k even, we show that there does not even exist a finite pseudofree G-CW complex $X \simeq S^{2k}$, with $X^G = \emptyset$. For all *odd* integers $k \geqslant 1$, such complexes do exist, for example, by taking the join of S^2 with the action given by $G \subset \mathrm{SO}(3)$ and a finite Swan complex for G (see [**3**, **9**]).

Remark 1.1. My interest in this question was prompted by the recent paper of Edmonds [**2**], where he proves this result for k even. Our methods seem rather different. The discussion by Edmonds in [**2**, 4.1] combined with Theorem A shows that there are no effective pseudofree dihedral actions on S^n, for $n > 2$, even if some elements of G are allowed to reverse orientation.

2. *The chain complex*

In this section, we let $G = D_p$ and suppose that X is a finite G-CW complex such that $X \simeq S^{2k}$, with $k > 0$, and $X^G = \emptyset$. We further assume that the G-action is pseudofree and induces the identity on homology. It follows from [**6**, Proposition 7.3] that every non-identity element of G fixes exactly two points. We assume that $X^G = \emptyset$ since this is a necessary condition for a locally

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linear, pseudo-free action on a sphere (by Milnor's theorem). Let $\mathbf{C} = \mathbf{C}(X^{?}))$ denote the chain complex of X over the orbit category $\mathbb{Z}\Gamma := \mathbb{Z}\,\mathrm{Or}_{\mathcal{F}}\,G$ with respect to the family $\mathcal{F}$ of all proper subgroups of G (see [**1**] or [**7**] for this theory). The notation means that $\mathbf{C}_i(G/U) = C_i(X^U)$, for $U \leqslant G$, and the action of $N_G(U)/U$ on $\mathbf{C}_i(G/U)$ induced by the G-action on X is expressed algebraically through the functorial properties of $\mathbf{C}$.

Our pseudofree assumption on the G-CW complex X implies that $\mathbf{C}_i(G/U) = 0$, if $U \neq 1$ is a non-trivial subgroup of G, and $i > 0$. Therefore,

$$H_i(\mathbf{C})(G/U) = 0 \quad \text{if } i > 0, \text{ for all } U \neq 1. \tag{1}$$

From the homology of S^{2k} we have

$$H_0(\mathbf{C})(G/1) = \mathbb{Z} \quad \text{and} \quad H_i(\mathbf{C})(G/1) = 0 \quad \text{for } i \neq 0, 2k. \tag{2}$$

In addition, since we assumed that G acts trivially on the homology of S^{2k}, we have

$$H_{2k}(\mathbf{C})(G/1) = \mathbb{Z}, \quad \text{with trivial } G\text{-action}. \tag{3}$$

Let $H = \langle a \rangle$ and $K = \langle b \rangle$ denote particular subgroups of G, of order p and 2, respectively. The orbit types give the chain group

$$\mathbf{C}_0 = \mathbb{Z}[G/H^{?}] \oplus \mathbb{Z}[G/K^{?}] \oplus \mathbb{Z}[G/K^{?}],$$

where $\mathbb{Z}[G/V^{?}]$ denotes the free right module over the orbit category with values

$$\mathbb{Z}[G/V^{?}](G/U) = \mathbb{Z}\,\mathrm{Map}_G(G/U, G/V),$$

for all proper subgroups $U \leqslant G$. In particular, the homology of the fixed sets is given by

$$H_0(\mathbf{C})(G/H) = \mathbb{Z}[G/H^{?}](G/H) = \mathbb{Z}[N_G(H)/H] = \mathbb{Z}[\mathbb{Z}/2] \tag{4}$$

and

$$H_0(\mathbf{C})(G/K) = (\mathbb{Z}[G/K^{?}](G/K))^2 = (\mathbb{Z}[N_G(K)/K])^2 = \mathbb{Z} \oplus \mathbb{Z}. \tag{5}$$

Definition 2.1. A finite $\mathbb{Z}\Gamma$-chain complex $\mathbf{C}$ of finitely generated free $\mathbb{Z}\Gamma$-modules, which satisfies the algebraic conditions (1)–(5), is called a *pseudofree $\mathbb{Z}\Gamma$-chain complex with the $\mathbb{Z}$-homology of S^{2k}*.

One example of such a complex arises from the standard orthogonal action $Y = S(V)$ of the dihedral group on S^2 (for G as a subgroup of SO(3)). The SO(3)-representation $V = W \oplus \mathbb{R}_-$ is the sum of the standard 2-dimensional real representation W (given by the action on a regular $2p$-gon in the plane) and the non-trivial 1-dimensional real representation $\mathbb{R}_-$. The chain complex $\mathbf{D} = \mathbf{C}(Y^{?})$ over the orbit category has the form

$$\begin{array}{ccccc}
\big(\mathbb{Z}[G/1^{?}]\big)^2 & \longrightarrow & \big(\mathbb{Z}[G/1^{?}]\big)^3 & \longrightarrow & \mathbb{Z}[G/H^{?}] \oplus \big(\mathbb{Z}[G/K^{?}]\big)^2 \\
\| & & \| & & \| \\
\mathbf{D}_2 & \longrightarrow & \mathbf{D}_1 & \longrightarrow & \mathbf{D}_0,
\end{array}$$

where $H_2(\mathbf{D}) = \underline{\mathbb{Z}}_0$ is the $\mathbb{Z}\Gamma$-module with value $\underline{\mathbb{Z}}_0(G/1) = \mathbb{Z}$, and zero otherwise. The module $H_0 := H_0(\mathbf{D})$ has value $H_0(G/1) = \mathbb{Z}$, and values at G/H and G/K as listed above. In general, for any pseudofree $\mathbb{Z}\Gamma$-chain complex $\mathbf{C}$ with the $\mathbb{Z}$-homology of S^{2k}, we have $H_{2k}(\mathbf{C}) = \underline{\mathbb{Z}}_0$ and $H_0(\mathbf{C}) = H_0(\mathbf{D})$.

Lemma 2.2. *Suppose that $\mathbf{C}$ is a pseudofree $\mathbb{Z}\Gamma$-chain complex with the $\mathbb{Z}$-homology of S^{2k}. Then the complex $\mathbf{C}$ is chain homotopy equivalent to a finite free $2k$-dimensional chain* *complex* $\mathbf{C}'$, *with* $\mathbf{C}'_i = \mathbf{C}_i$ *for* $i \geqslant 4$, *whose initial part* $\mathbf{C}'_2 \to \mathbf{C}'_1 \to \mathbf{C}'_0$ *is chain isomorphic to* $\mathbf{D}$.

Proof. Since $H_0(\mathbf{C}) = H_0(\mathbf{D})$, this follows from the version of Schanuel's lemma over the orbit category given in the proof of [**4**, Lemma 8.12]. □

An immediate consequence is the statement of Theorem A for k even.

Corollary 2.3 (Edmonds [**2**]). *Let* $G = D_p$. *If* k *is even, there is no effective pseudofree* G*-action on a finite* G-CW *complex* $X \simeq S^{2k}$, *inducing the identity on homology.*

Proof. Let $\mathbf{C} = \mathbf{C}(X^?)$ denote the chain complex over the orbit category of such an action. From the chain equivalent complex $\mathbf{C}' \simeq \mathbf{C}$, we can extract a periodic resolution

$$0 \longrightarrow \underline{\mathbb{Z}}_0 \longrightarrow \mathbf{C}_{2k} \longrightarrow \mathbf{C}_{2k-1} \longrightarrow \cdots \longrightarrow \mathbf{C}_4 \longrightarrow \mathbf{C}''_3 \longrightarrow \underline{\mathbb{Z}}_0 \longrightarrow 0$$

since $H_2(\mathbf{D}) = H_{2k}(\mathbf{C}) = \underline{\mathbb{Z}}_0$, where $\mathbf{C}''_3$ is a direct sum of copies of $\mathbb{Z}[G/1^{?}]$. By evaluating at $G/1$, we obtain a periodic projective resolution from $\mathbb{Z}$ to $\mathbb{Z}$ over $\mathbb{Z}G$ of length $(2k-2)$. Since $G = D_p$ has periodic cohomology of period 4 (and not 2), we conclude that k is odd. □

Proof of Theorem A (k *odd*). Suppose, if possible, that we have a locally linear and orientation-preserving pseudofree topological action of G on S^{2k}, for some odd integer $k \geqslant 3$. Then there exists a finite G-CW complex $X \simeq S^{2k}$, and a chain homotopy equivalence $\mathbf{C}(X^?) \simeq \mathbf{C}'$ provided by Lemma 2.2. We may identify the singular set $\mathrm{Sing}(X)$ of X with the singular set of the given action on S^{2k}. Let $\{x_0, x_1, x_2\} \subset \mathrm{Sing}(X)$ denote representatives of the distinct G-orbits of singular points (with $G_{x_0} = H$, and $G_{x_i} = K$ for $i = 1, 2$). Around each singular point x_i, with $0 \leqslant i \leqslant 2$, we can choose a linearly embedded 2-disk slice $G \times_{G_{x_i}} D^2 \subset S^{2k}$, since the action (S^{2k}, G) is locally linear. This gives a G-equivariant embedding

$$f_0 : \bigcup_{0 \leqslant i \leqslant 2} (G \times_{G_{x_i}} D^2) \subset S^{2k}.$$

Since the pseudofree orbit structure of the standard G-action on $S^2 = S(V)$ is the same for any locally linear action on S^{2k}, we can consider f_0 to be a G-equivariant embedding of a tubular neighborhood of the singular set of $S(V)$ into S^{2k}. By obstruction theory, and since $k \geqslant 3$, we can extend this embedding f_0 to a G-equivariant embedding $f : S(V) \subset S^{2k}$. Non-equivariantly such an embedding of $S^2 \subset S^{2k}$ is isotopic to a standard embedding. We have thus obtained a dihedral action on S^{2k} of the type considered in my earlier joint work with Pedersen [**5**], namely, one conjugate to 'a topological action on a sphere which is free off a standard proper subsphere, and given by a $S(V)$ on the subsphere'. However, we proved in [**5**, Theorem 7.11] that such an action exists if and only if the representation V on the subsphere contains two $\mathbb{R}_-$ factors. Since this is not the case for the standard SO(3)-representation V of G, we conclude that a pseudofree G-action on S^{2k} does not exist for $k > 1$. □

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Ian Hambleton
Department of Mathematics & Statistics
McMaster University
Hamilton, ON
Canada L8S 4K1

hambleton@mcmaster.ca